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# On Additive Weight Approximation and Simultaneous Chebyshev Approximation Using Varisolvent Families

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The purpose of this paper is to study the problem of Chebyshev approximation with an additive weight and the equivalent problem of simultaneous Chebyshev approximation, when the approximating elements are drawn from a nonlinear family. Precise definitions and statements are given below.

DEFINITION 1. Let $\mathscr{F}$ be a set of bounded real valued functions on the interval $[a, b]$ ($a < b$). Let $\phi$ and $W$ be bounded real valued functions on $[a, b]$, with $W \geqslant 0$ on $[a, b]$. The function $F$, in $\mathscr{F}$, is a *best approximation to $\phi$ from $\mathscr{F}$ with the additive weight $W$* if $\sup\{|F(x) - \phi(x)| + W(x): x \text{ in } [a, b]\} \leqslant \sup\{|G(x) - \phi(x)| + W(x): x \text{ in } [a, b]\}$ for all $G$ in $\mathscr{F}$. To state it more simply, if $F$ satisfies the above definition, we will say that *$F$ is a best additive weight approximation to $(\phi, W)$ in $\mathscr{F}$*.

DEFINITION 2. Let $\mathscr{F}$ be a set of bounded real valued functions on the interval $[a, b]$ ($a < b$). Let $S$ be a uniformly bounded set of real valued functions on $[a, b]$. The function $F \in \mathscr{F}$ is a *best simultaneous approximation to the functions in $S$ from $\mathscr{F}$ if* $\sup_{\phi\in S} \sup\{|F(x) - \phi(x)|: x \text{ in } [a, b]\} \leqslant \sup_{\phi\in S} \sup\{|G(x) - \phi(x)|: x \text{ in } [a, b]\}$ for all $G$ in $\mathscr{F}$.

It was observed [2] that if $F$, $\psi_1$ and $\psi_2$ are bounded real valued functions on $[a, b]$, then

$$\max\{\sup_{x\in[a,b]} |F(x) - \psi_1(x)|, \sup_{x\in[a,b]} |F(x) - \psi_2(x)|\}$$
$$= \sup_{x\in[a,b]} \{|F(x) - \tfrac{1}{2}(\psi_1(x) + \psi_2(x))| + \tfrac{1}{2}|\psi_1(x) - \psi_2(x)|\}.$$

Therefore the following two theorems are true.

THEOREM 3 [2]. *Let $\mathscr{F}$ be a subset of $C[a, b]$, and let $\psi_1$ and $\psi_2$ be bounded real valued functions on $[a, b]$. If $F$ in $\mathscr{F}$ is a best simultaneous approximation to the functions in $\{\psi_1, \psi_2\}$ then $F$ is a bets a additive weight approximation to $(\phi, W)$ where $\phi = \frac{1}{2}(\psi_1 + \psi_2)$ and $W = |\frac{1}{2}(\psi_1 - \psi_2)|$.*

THEOREM 4 [2]. *Let $\mathscr{F}$ be a subset of $C[a, b]$, let $\phi$ and $W$ be bounded real valued functions on $[a, b]$ with $W \geqslant 0$ on $[a, b]$. If $F$ in $\mathscr{F}$ is a best additive weight approximation to $(\phi, W)$ in $\mathscr{F}$, then $F$ is a best simultaneous approximation to the functions in $\{\psi_1, \psi_2\}$ where $\psi_1 = \phi - W$ and $\psi_2 = \phi + W$.*

Let $S$ be a uniformly bounded set of real valued functions on $[a, b]$ and let $\mathscr{F} \subset C[a, b]$. According to [1], there exist two upper semicontinuous functions $-\phi_1$ and $\phi_2$ on $[a, b]$ such that $\phi_1 \leqslant \phi_2$ and if $F$ is in $\mathscr{F}$, then $F$ is a best simultaneous approximation to the functions in $S$ if and only if $F$ is a best simultaneous approximation to the functions in $\{\phi_1, \phi_2\}$. Thus, when one considers simultaneous approximation, it suffices to assume $S$ consists of two functions, $\phi_1$ and $\phi_2$ where $\phi_1 \leqslant \phi_2$ and $-\phi_1$ and $\phi_2$ are upper semicontinuous on $[a, b]$. By using this result, when one considers the additive weight problem with bounded real valued functions $\phi$ and $W$ on $[a, b]$, $W \geqslant 0$ on $[a, b]$, it suffices to assume that the functions $W - \phi$ and $W + \phi$ are upper semicontinuous on $[a, b]$.

We now discuss the additive weight approximation problem noting that all the definitions and results may be translated into definitions and results for the simultaneous approximation problem

Let $\psi$ be a bounded real valued function on $[a, b]$, then $\| \psi \|$ is defined as $\| \psi \| = \sup\{| \psi(x)| : x \text{ in } [a, b]\}$. The function $\psi$ is said to be upper semicontinuous on $[a, b]$, if $\psi(x) \geqslant \lim_{y \to x} \psi(y)$ for all $x$ in $[a, b]$. The function $\psi$ is lower semicontinuous if and only if the function $-\psi$ is upper semicontinuous.

DEFINITION 5. Let $\phi$ and $W$ be real valued functions on $[a, b]$ such that $W + \phi$ and $W - \phi$ are upper semicontinuous on $[a, b]$ and $W \geqslant 0$ on $[a, b]$. Let $F$ be a continuous function on $[a, b]$.

(1) *$F$ has a straddle point, $x$, with respect to $(\phi, W)$* if $F(x) - \phi(x) = 0$ and $W(x) = \| \, | F - \phi | + W \|$. (When this is translated into the simultaneous approximation problem, Dunham's definition of a straddle point is obtained [3].)

(2) *$F$ alternates $n$ times $(n \geqslant 0)$ with respect to $(\phi, W)$* if there exist $\{x_i\}_{i=1}^{n+1} \subset [a, b]$, $x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$ such that

(a) $| F(x_i) - \phi(x_i)| + W(x_i) = \| \, | F - \phi | + W \|$ $(i = 1, \ldots, n + 1)$

(b) $(F(x_i) - \phi(x_i))(F(x_{i+1}) - \phi(x_{i+1})) < 0$ $(1 \leqslant i \leqslant n)$.

(3) In 2 above the set $\{x_i\}_{i=1}^{n+1}$ is called a set of *alternation points of F with respect to* $(\phi, W)$.

(4) *F has a constant error with respect to* $(\phi, W)$ if

$$|F(x) - \phi(x)| + W(x) = \| |F - \phi| + W \| \qquad \text{for all } x \text{ in } [a, b].$$

In what follows the family $\mathscr{F}$ will be assumed to be a varisolvent family as defined in [4]. The necessary definitions and observations concerning varisolvency are given next for completeness.

DEFINITION 6. Let $\{I_i\}_{i=1}^{n}$ be a sequence of closed intervals $(n \geqslant 1)$. The sequence $\{I_i\}_{i=1}^{n}$ will be called an *increasing sequence of closed intervals* if for every $x$ in $I_i$ and every $y$ in $I_{i+1}$ $(1 \leqslant i < n)$, it is true that $x < y$.

DEFINITION 7. Let $\mathscr{F}$ be a family of functions in $C[a, b]$ and let $F$ be in $\mathscr{F}$. The ordered pair of integers $(n_1, n_2)$ with $n_1 \geqslant 0$ and $n_2 \geqslant 1$ is *a degree of F with respect to* $\mathscr{F}$ if the following conditions are met:

(1) Let $\epsilon > 0$ and $\sigma$ in $\{-1, 1\}$ be arbitrarily chosen. If $n_1 = 1$, then there is a function, $G$, in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)(F(x) - G(x)) > 0$ on $[a, b]$. (The factor $(-1)$ is superfluous for this part of the definition.) If $n_1 > 1$, if $\delta$ is an arbitrary element of $\{0, 1\}$, and if $\{[c_i, d_i]\}_{i=1}^{n_1-\delta}$ is an arbitrary increasing sequence of closed intervals where $c_1 = a$ and $d_{n_1-\delta} = b$, then there is a function, $G$, in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ $(i = 1,..., n_1 - \delta)$.

(2) If $G$ is a continuous function on $[a, b]$ and $a \leqslant x_1 < \cdots < x_{n_2+1} \leqslant b$ such that $(F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) < 0$ $(i = 1,..., n_2)$, then $G$ is not in the family.

It is noted that $n_1 = 0$ is permissible and that if $(0, n_2)$ is a degree of $F$ with respect to $\mathscr{F}$, only the integer $n_2$ gives any information about the function's relation to the rest of the family. Furthermore, if $F$ has $(n_1, n_2)$ as a degree, $(0, n_2)$ is also a degree.

What the above definition is saying is that if the function, $F$, in $\mathscr{F}$ has $(n_1, n_2)$ as a degree with respect to $\mathscr{F}$, then there is a function $G$ in $\mathscr{F}$ that is arbitrarily close to $F$ on $[a, b]$ such that $F - G$ alternates in sign on $n_1$ $(n_1 - 1)$ intervals. Furthermore, every member of $\mathscr{F}$ that is distinct from $F$ crosses $F$ at most $n_2 - 1$ times in $(a, b)$. If an approximating family, $\mathscr{F}$, satisfies Rice's definition of Property $A$ [5], then the first part of Definition 7 would be satisfied, but the converse is not necessarily true.

*Remark* 8. If $(n_1, n_2)$ is a degree of $F$ with respect to $\mathscr{F}$, then $n_1 \leqslant n_2$. The definition seems to indicate that a function is permitted to have more than one degree. This is, in fact, the case. If $F$ has a degree $(n_1, n_2)$ with respect to $\mathscr{F}$, the following lemma and corollary, which appeared in [4], gives some information as to what other degrees $F$ may have.

LEMMA 9. *If $F$ belongs to $\mathscr{F}$ and has degree $(n_1, n_2)$ with respect to $\mathscr{F}$, then*

(1) *$(n_1 - 1, n_2)$ is also a degree of $F$ with respect to $\mathscr{F}$ as long as $n_1$ is not zero or three*;

(2) *$(n_1, n_2 + 1)$ is also a degree of $F$ with respect to $\mathscr{F}$.*

COROLLARY 10. *If $F$ in $\mathscr{F}$ has a degree $(n_1, n_2)$ with respect to $\mathscr{F}$ and $n_1 \leqslant 3$, then $(m_1, m_2)$ is also a degree where $3 \leqslant m_1 \leqslant n_1$ and $n_2 \leqslant m_2 < \infty$.*

The next four lemmas give sufficient conditions for the existence of a better additive weight approximation to $(\phi, W)$ than $F$ from $\mathscr{F}$, where $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$.

LEMMA 11. *Let $\mathscr{F} \subset C[a, b]$. Let $F$ in $\mathscr{F}$ have $(n_1, n_2)$ as a degree with respect to $\mathscr{F}$ where $n_1 \geqslant 2$. Let $\phi$ and $W$ be real valued functions on $[a, b]$ such that $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$. Suppose $F$ alternates $n_1 - 1$ times but does not alternate $n_1$ times with respect to $(\phi, W)$. If $F$ has no straddle points with respect to $(\phi, W)$, then there exist $G$ in $\mathscr{F}$ such that $\| \, | G - \phi | + W \| < \rho = \| \, | F - \phi | + W \|$.*

*Proof.* Let $\{x_i\}_{i=1}^{n_1}$ be a set of alternation points. Define $x_0 = a_1$ and $x_{n_1+1} = b$ and let $\sigma$ belong to $\{-1, 1\}$ such that $\sigma(-1)(F(x_1) - \phi(x_1)) + W(x_1) = \rho$. Since $W - \phi\,(W + \phi)$ is upper semicontinuous and $F\,(-F)$ is continuous, it follows that $F - \phi + W$ $(\phi - F + W)$ is upper semicontinuous. Therefore, the sets $\{x \text{ in } [a, b]\colon F(x) - \phi(x) + W(x) = \rho\}$ and $\{x \text{ in } [a, b]\colon \phi(x) - F(x) + W(x) = \rho\}$ are closed and nonempty. Thus, the values $x_i^L$ and $x_i^U$ $(i = 1,..., n_1)$ defined below exist. Let $x_i^L = \min\{x \text{ in } [x_{i-1}, x_i]\colon \sigma(-1)^i(F(x) - \phi(x)) + W(x) = \rho\}$ and $x_i^U = \min\{x \text{ in } [x_i, x_{i+1}]\colon \sigma(-1)^i(F(x) - \phi(x)) + W(x) = \rho\}$ $(i = 1,..., n_1)$. Since $x_i^U = x_{i+1}^L$ for some $i$ $(1 \leqslant i < n_1)$ would imply that $x_i^U$ is a straddle point of $F$ with respect to $(\phi, W)$ and since $x_i^U > x_{i+1}^L$ for some $i$ $(1 \leqslant i < n_1)$ would imply that $F$ alternates $n_1 + 1$ times with respect to $(\phi, W)$, we have that $x_i^U < x_{1+i}^L$ $(i = 1,..., n_1 - 1)$.

Define $\mu = \frac{1}{3}\min\{x_{i+1}^L - x_i^U\colon i = 1,...,n_1 - 1\}$ and define $I_1 = [a, x_1^U + \mu]$, $I_i = [x_i^L - \mu, x_i^U + \mu]$ $(1 < i < n_1)$ and $I_{n_1} = [x_{n_1}^L - \mu, b]$.

Let $\epsilon_1 = \min_{i=1,...,n_1} \min\{\sigma(-1)^i(F(x) - \phi(x)) + \rho - W(x)\colon x \text{ in } I_i\}$. It follows from the definition of $\rho$ that $\epsilon_1 \geqslant 0$. Since $\sigma(-1)^i(F(x) - \phi(x) + \rho - W(x)$ is lower semicontinuous, there exists a $y_i$ in $I_i$ such that

$\sigma(-1)^i(F(y_i) - \phi(y_i)) + \rho - W(y_i) = \min\{\sigma(-1)^i(F(x) - \phi(x)) + \rho - W(x)$: $x$ in $I_i\}$ $(i = 1,..., n_1)$. If $\epsilon_1 = 0$, then for some $i$ $(1 \leqslant i \leqslant n_1)$, $\sigma(-1)^i(F(y_i) - \phi(y_i)) + \rho - W(y_i) = 0$. By the construction on $[x_i^L, x_i^U]$, we have that $y_i$ belongs to $I_i - [x_i^L, x_i^U]$. If $y_i$ belongs to $[x_i^L - \mu, x_i^L)$, then by construction, $y_i$ is also contained in $[x^U_{i-1}, x^U_{i-}]$ which is a contradiction. Similarly, there is a contradiction if $y_i$ belongs to $(x_i^U, x_i^U + \mu]$. Thus $\epsilon_1 > 0$.

It is noted that $\sup\{|F(x) - \phi(x)| + W(x)$: $x$ in $[a, b] - \bigcup_{i=1}^{n_1} I_i\} < \rho$. To see this, assume that $\sup\{|F(x) - \phi(x)| + W(x)$: $x$ in $[a, b] - \bigcup_{i=1}^{n_1} I_i\} = \rho$. Then there exists a sequence $\{y_j\}_{j=1}^{\infty}$ contained in $[a, b] - \bigcup_{i=1}^{n_1} I_i$ such that $\rho = \lim\sup_{j\to\infty}(|F(y_j) - \phi(y_j)| + W(y_j))$. Without loss of generality, we assume that $\{y_j\}_{j=1}^{\infty}$ converges to $y_0$ in $[a, b]$. By upper semicontinuity of $|F - \phi| + W$, we have that $|F(y_0) - \phi(y_0)| + W(y_0) \geqslant \rho$. Since $\| |F - \phi| + W\| = \rho$, it follows that $|F(y_0) - \phi(y_0)| + W(y_0) = \rho$ belongs to $\bigcup_{i=1}^{n_1} [x_i^L, x_i^U]$ which is a contradiction. Therefore $\epsilon_2$ is positive, where $\epsilon_2 = \rho - \sup\{|F(x) - \phi(x)| + W(x)$: $x$ in $[a, b] - \bigcup_{i=1}^{n_1} I_i\}$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Since $F$ has $(n_1, n_2)$ as a degree, there exist $G$ in $\mathscr{F}$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) \geqslant 0$ on $I_i$ $(i = 1,..., n_1)$.

We now show that $\| |G - \phi| + W\| < \| |F - \phi| + W\|$. Since $|G - \phi| + W$ is upper semicontinuous, it suffices to show that $|G(x) - \phi(x)| + W(x) < \rho$ for all $x$ in $[a, b]$. If $x$ belongs to $[a, b] - \bigcup_{i=1}^{n_1} I_i$, then $0 \leqslant |G(x) - \phi(x)| + W(x) \leqslant |G(x) - F(x)| + |F(x) - \phi(x)| + W(x) < \epsilon + \sup\{|F(x) - \phi(x)| + W(x)$: $x$ in $[a, b] - \bigcup_{i=1}^{n_1} I_i\} = \epsilon + \rho - \epsilon_2 \leqslant \rho$. Thus $|G(x) - \phi(x)| + W(x) < \rho$ on $[a, b] - \bigcup_{i=1}^{n_1} I_i$. If $x$ belongs to $I_i$ for some $i$ $(1 \leqslant i \leqslant n_1)$, then

$$-\epsilon < \sigma(-1)^i(G(x) - F(x)) < 0$$

and

$$-\rho + W(x) + \epsilon_1 \leqslant \sigma(-1)^i(F(x) - \phi(x)) \leqslant \rho - W(x).$$

By adding these inequalities, we have

$$\begin{aligned} -\rho + W(x) + (\epsilon_1 - \epsilon) &< \sigma(-1)^i(G(x) - \phi(x)) \\ &< \rho - W(x) \qquad \text{for all } x \text{ in } I_i. \end{aligned}$$

This implies that $|G(x) - \phi(x)| < \rho - W(x)$ for $x$ in $I_i$, or $|G(x) - \phi(x)| + W(x) < \rho$ for all $x$ in $I_i$. This completes the proof of the lemma.

LEMMA 12. *Let $\mathscr{F} \subset C[a, b]$. Let $F$ in $\mathscr{F}$ have degree $(3, n_2)$ with respect to $\mathscr{F}$. Let $\phi$ and $W$ be real valued functions on $[a, b]$ such that $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$. Suppose $F$ alternates once, but not twice, with respect to $(\phi, W)$. If $F$ has no straddle points with* *respect to* $(\phi, W)$, *then there exist* $G$ *in* $\mathscr{F}$ *such that* $\| \, | G - \phi | + W \| < \| \, | F - \phi | + W \|$.

*Proof.* The proof is the same as the preceeding proof with $n_1$ (in the proof) replaced by 2, (instead of 3).

By summarizing the above two lemmas, and by using Corollary 10, we have the following corollary.

COROLLARY 13. *Let* $\mathscr{F} \subset C[a, b]$. *Let* $F$ *have a degree* $(n_1, n_2)$ *with respect to* $\mathscr{F}$, $(n_1 \geqslant 2)$. *Let* $\phi$ *and* $W$ *be real valued functions on* $[a, b]$ *such that* $W - \phi$ *and* $W + \phi$ *are upper semicontinuous and* $W \geqslant 0$ *on* $[a, b]$. *Suppose* $F$ *alternates once, but does not alternate* $n_1$ *times, with respect to* $(\phi, W)$. *If* $F$ *has no straddle points with respect to* $(\phi, W)$, *then there exist* $G$ *in* $\mathscr{F}$ *such that* $\| \, | G - \phi | + W \| < \| \, | F - \phi | + W \|$.

LEMMA 14. *Let* $\mathscr{F} \subset C[a, b]$. *Let* $F$ *in* $\mathscr{F}$ *have a degree* $(3, n_2)$ *with respect to* $\mathscr{F}$. *Let* $\phi$ *and* $W$ *be real valued functions on* $[a, b]$ *such that* $W - \phi$ *and* $W + \phi$ *are upper semicontinuous and* $W \geqslant 0$ *on* $[a, b]$. *Suppose* $F$ *does not alternate once with respect to* $(\phi, W)$, *suppose* $F$ *does not have a straddle point with respect to* $(\phi, W)$ *and suppose* $F$ *does not have a constant error with respect to* $(\phi, W)$. *Then, there exist* $G$ *in* $\mathscr{F}$ *such that* $\| \, | G - \phi | + W \| < \rho = \| \, | F - \phi | + W \|$.

*Proof.* Since $| F - \phi | + W$ is upper semicontinuous, there exists a $z$ in $[a, b]$ such that $| F(z) - \phi(z)| + W(z) = \rho$. Let $\sigma$ belong to $\{-1, 1\}$ such that $\sigma(-1)(F(z) - \phi(z)) + W(z) = \rho$. The value $\sigma$ is well defined since $F$ does not alternate once with respect to $(\phi, W)$ and $F$ does not have a straddle point with respect to $(\phi, W)$. By upper semicontinuity, the set $\{x \text{ in } (a, b)\colon \sigma(-1)(F(x) - \phi(x)) + W(x) < \rho\}$ is an open set. Let $y$ belong to $\{x \text{ in } (a, b)\colon \sigma(-1)(F(x) - \phi(x)) + W(x) < \rho\}$ and let $a_1$ and $b_1$ be such that $a \leqslant a_1 < y < b_1 \leqslant b$ and $\sigma(-1)(F(x) - \phi(x)) + W(x) < \rho$ for all $x$ in $[a_1, b_1]$. The set $[a_1, b_1]$ is not empty by the semicontinuity assumptions and by the assumption that $F$ does not have a constant error with respect to $(\phi, W)$. Define $I_1 = [a, a_1]$, $I_2 = [a_1 + \frac{1}{3}(b_1 - a_1), b_1 - \frac{1}{3}(b_1 - a_1)]$, and $I_3 = [b_1, b]$.

Let $\epsilon_1 = \min\{\sigma(-1)^i(F(x) - \phi(x)) + \rho - W(x)\colon x \text{ in } I_i \ (i = 1, 3)\}$. The lower semicontinuity of the function $\sigma(-1)^i(F - \phi) - W + \rho$ ensures the existence of $\epsilon_1$. By the definition of $\rho$, we have $\epsilon_1 \geqslant 0$. Assume $\epsilon_1 = 0$. Then there exists a $z$ in $I_1 \cup I_3$ such that $-\rho + W(z) = \sigma(-1)(F(z) - \phi(z))$ or $\rho = \sigma(F(z) - \phi(z)) + W(z)$. This implies that $F$ has one alternation with respect to $(\phi, W)$ if $z \neq x$, or $F$ has a straddle point with respect to $(\phi, W)$ if $z = x$. Therefore, $\epsilon_1$ is positive.

Let $\epsilon_2 = \rho - \sup\{| F(x) - \phi(x)| + W(x)\colon x \text{ in } (a_1, b_1)\} > 0$ and let $\epsilon =$

$\min(\epsilon_1, \epsilon_2)$. Since $F$ has $(3, n_2)$ as a degree, there exist $G$ in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on $I_i$ $(i = 1, 2, 3)$.

We now show that $\| \, | G - \phi | + W \| < \rho$. Since $| G - \phi | + W$ is upper semicontinuous on $[a, b]$, it suffices to show that $| G(x) - \phi(x)| + W(x) < \rho$ for all $x$ in $[a, b]$. If $x$ belongs to $(a_1, b_1)$ we have $| G(x) - \phi(x)| + W(x) \leqslant | G(x) - F(x)| + | F(x) - \phi(x)| + W(x) < \epsilon + \sup\{| F(x) - \phi(x)| + W(x): x \text{ in } (a_1, b_1)\} = \epsilon + \rho - \epsilon_2 \leqslant \rho$. If $x$ belongs to $I_i$ $(i = 1, 3)$, then

$$-\epsilon < \sigma(-1)^i(G(x) - F(x)) < 0$$

and

$$-\rho + W(x) + \epsilon_1 \leqslant \sigma(-1)^i(F(x) - \phi(x)) \leqslant \rho - W(x).$$

By adding these two inequalities, we have

$$\begin{aligned} -\rho + W(x) + \epsilon_1 - \epsilon < \sigma(-1)^i(G(x) - \phi(x)) \\ < \rho - W(x) \qquad \text{for all } x \text{ in } I_i, \end{aligned}$$

$(i = 1, 3)$. Thus $| G(x) - \phi(x)| < \rho - W(x)$ for all $x$ in $I_1 \cup I_3$. This completes the proof of the lemma.

LEMMA 15. *Let $\mathscr{F} \subset C[a, b]$. Let $F$ in $\mathscr{F}$ have a degree $(n_1, n_2)$ with respect to $\mathscr{F}$ $(n_1 = 1$ or $n_1 = 2)$. Let $\phi$ and $W$ be real valued functions on $[a, b]$ such that $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$. Suppose $F$ has no straddle points with respect to $(\phi, W)$, and suppose $F$ does not alternate once with respect to $(\phi, W)$. Then, there exist $G$ in $\mathscr{F}$ such that $\| \, | G - \phi | + W \| < \rho = \| \, | F - \phi | + W \|$.*

*Proof.* Let $\epsilon_1 = \min\{\sigma(-1)(F(x) - \phi(x)) + \rho - W(x): x \text{ in } [a, b]\}$ where $\sigma$ belongs to $\{-1, 1\}$ such that for some $y$ in $[a, b]$, $\sigma(-1)(F(y) - \phi(y)) + W(y) = \rho$. By the semicontinuity assumption on $W + \phi$ and $W - \phi$, such a $y$ does exist; $\sigma$ is well defined because of the assumption that there are no straddle points; $\epsilon_1 > 0$ since it is assumed that $F$ does not alternate once with respect to $(\phi, W)$. Since $(n_1, n_2)$ is a degree of $F$, there exist $G$ in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)(F(x) - G(x)) > 0$ on $[a, b]$. $G$ is the desired function.

We now give a necessary condition for a function, $F$, in $\mathscr{F}$ to be a best additive weight approximation to $(\phi, W)$ where $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$.

THEOREM 16. *Let $\mathscr{F} \subset C[a, b]$. Let $F$ in $\mathscr{F}$ have a degree $(n_1, n_2)$ with respect to $\mathscr{F}$. Let $\phi$ and $W$ be real valued functions on $[a, b]$ where $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$.*

*Suppose* $\| \, |F - \phi| + W \| \leqslant \| \, |G - \phi| + W \|$ *for all* $G$ *in* $\mathscr{F}$. *Then, one of the following is true.*

(1) *$F$ has a straddle point with respect to* $(\phi, W)$.

(2) *$F$ alternates* $n_1$ *times with respect to* $(\phi, W)$.

(3) *$F$ has a constant error with respect to* $(\phi, W)$.

*Proof.* If $n_1 = 0$, the upper semicontinuity of $|F - \phi| + W$ implies that there exists an $x_1$ in $[a, b]$ such that $|F(x_1) - \phi(x_1)| + W(x_1) = \| \, |F - \phi| + W \|$; thus statement two is satisfied. Assume $n_1 \geqslant 1$ and assume statements one, two, and three are not true. Then the previous lemmas ensure the existence of $G$ in $\mathscr{F}$ such that $\| \, |G - \phi| + W \| < \| \, |F - \phi| + W \|$, which is a contradiction.

We now give a sufficiency condition for a function, $F$, in $\mathscr{F}$ to be a best additive weight approximation to $(\phi, W)$ where $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$.

THEOREM 17. *Let* $\mathscr{F} \subset C[a, b]$. *Let* $F$ *in* $\mathscr{F}$ *have a degree* $(n_1, n_2)$ *with respect to* $\mathscr{F}$. *Let* $\phi$ *and* $W$ *be real valued functions on* $[a, b]$ *where* $W - \phi$ *and* $W + \phi$ *are upper semicontinuous and* $W \geqslant 0$ *on* $[a, b]$. *Suppose* $F$ *alternates* $n_2$ *times with respect to* $(\phi, W)$. *Then,* $\| \, |F - \phi| + W \| \leqslant \| \, |G - \phi| + W \|$ *for all* $G$ *in* $\mathscr{F}$.

*Proof.* If $F$ has a straddle point, $x$, with respect to $(\phi, W)$, then $\| \, |F - \phi| + W \| = |F(x) - \phi(x)| + W(x) = W(x) \leqslant |G(x) - \phi(x)| + W(x) \leqslant \| \, |G - \phi| + W \|$, for all $G$ in $\mathscr{F}$.

Assume $F$ does not have a straddle point with respect to $(\phi, W)$ and let $G$ belong to $C[a, b]$ such that $\| \, |G - \phi| + W \| < \rho = \| \, |F - \phi| + W \|$. Let $\{x_i\}_{i=1}^{n_2+1}$ be a set of $n_2 + 1$ alternation points of $F$ with respect to $(\phi, W)$. Let $\sigma$ belong to $\{-1, 1\}$ such that $\sigma(-1)(F(x_1) - \phi(x_1)) + W(x_1) = \rho$. Then $\sigma(-1)^i(F(x_i) - \phi(x_i)) + W(x_i) = \rho$ $(i = 1, \ldots, n_2 + 1)$. Observe that

$$\begin{aligned}
\sigma(-1)^i(F(x_i) - G(x_i)) &= \sigma(-1)^i(F(x_i) - \phi(x_i)) + \sigma(-1)^i(\phi(x_i) - G(x_i)) \\
&= (\rho - W(x_i)) + \sigma(-1)^i(\phi(x_i) - G(x_i)) \\
&= \rho - [\sigma(-1)^i(G(x_i) - \phi(x_i)) + W(x_i)] \\
&> 0 \qquad (i = 1, \ldots, n_2 + 1).
\end{aligned}$$

Therefore

$$[\sigma(-1)^i(F(x_i) - G(x_i))][\sigma(-1)^{i+1}(F(x_{i+1}) - G(x_{i+1}))] > 0 \quad (i = 1, \ldots, n_2)$$

or

$$(F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) < 0 \qquad (i = 1, \ldots, n_2).$$

Thus, $G$ does not belong to $\mathscr{F}$.

Summarizing this section for $\mathscr{F}$ a varisolvent family of functions on $[a, b]$, we have the following theorem.

THEOREM 18. *Let $\mathscr{F}$ be a varisolvent family of functions on $[a, b]$, let $F$ in $\mathscr{F}$ have $(n_1, n_2)$ as a degree with respect to $\mathscr{F}$, and let $\phi$ and $W$ be real valued functions on $[a, b]$ where $W - \phi$ and $W + \phi$ are upper semicontinuous and $W \geqslant 0$ on $[a, b]$.*

(1) *If $\| \, | F - \phi | + W \| \leqslant \| \, | G - \phi | + W \|$ for all $G$ in $\mathscr{F}$, then either $F$ has a straddle point, alternates $n_1$ times, or has a constant error with respect to $(\phi, W)$.*

(2) *If $F$ alternates $n_2$ times with respect to $(\phi, W)$, then*

$$\| \, | F - \phi | + W \| \leqslant \| \, | G - \phi | + W \| \qquad \text{for all } G \text{ in } \mathscr{F}.$$

It is noted that if $\mathscr{F}$ is a unisolvent family of degree $n$ on $[a, b]$, then the above theory reduces to that of Dunham's [3] for the simultaneous approximation problem.

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